

# DESINGULARIZATION OF LEGENDRIAN SURFACES - THE QUASI-ORDINARY CASE

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ABSTRACT. In this paper we prove a desingularization theorem for Legendrian surfaces that are the conormal of a quasi-ordinary hypersurface.

## 1. INTRODUCTION

The main purpose of this paper is to prove a desingularization theorem for Lagrangean surfaces of a contact manifold of dimension 5. For the moment we limit ourselves to considering the quasi-ordinary case: the case when the Lagrangean variety is the conormal of a quasi-ordinary surface. Our proof depends on recent work on the computation of the limits of tangents of a quasi-ordinary surface [1]. At the moment we do not have a systematic way to compute the limits of tangents of a general class of hypersurfaces. Once this problem is solved, it should not be very hard to generalize our main result to an arbitrary Lagrangean variety. Our paper generalizes previous work on the desingularization of Lagrangean curves (see [8]). Lipman proved a desingularization theorem for quasi-ordinary surfaces (see [7]). Ban and MacEwan [2] showed that Lipman's algorithm produces an embedded desingularization, which coincides with the algorithm of Bierston and Milman [3].

We show that when a quasi-ordinary surface  $S$  has trivial limits of tangents and  $L$  is an admissible center for  $S$ , the conormal of the blow up of  $S$  along  $L$  equals the blow up of the conormal of  $S$  along the conormal of  $L$ .

We recall that each Lagrangean variety of  $\mathbb{P}^*X$  is the conormal of its projection on  $X$ . Moreover, each Lagrangean variety is isomorphic to the conormal of a hypersurface with trivial limits of tangents (see [4]). Hence we can apply the procedure to each germ of Lagrangean surface.

One of the main motivations of this work is its application to the desingularization of certain classes of holonomic systems of partial differential equations (see [10]).

## 2. LOGARITHMIC CONTACT MANIFOLDS

All manifolds considered in this paper are complex analytic manifolds.

A subset  $Y$  of a manifold  $X$  is called a divisor with normal crossings at  $o \in X$  if there is an open neighborhood  $U$  of  $o$ , a system of local coordinates  $(x_1, \dots, x_n)$  and a nonnegative integer  $\nu$  such that  $x_i(o) = 0, i = 1, \dots, n$ , and

$$(2.1) \quad Y \cap U = \{x_1 \cdots x_\nu = 0\}.$$

We call  $\nu$  the index of  $Y$  at  $o$ . We say that  $Y$  is a *divisor with normal crossings* if  $Y$  is a divisor with normal crossings at each point of  $X$ .

Let  $Y_1, \dots, Y_\nu$  be the irreducible components of  $Y$  in a neighborhood of  $o$ . Let  $1 \leq \mu \leq \nu$ . Set  $Z = Y_1 \cap \dots \cap Y_\mu$ . We call  $(Y_{\mu+1} \cup \dots \cup Y_\nu) \cap Z$  the *normal crossings divisor induced in  $Z$  by  $Y$* .

Let  $j : X \setminus Y \hookrightarrow X$  be the open inclusion. Let  $\mathcal{O}_X$  denote the sheaf of holomorphic functions of  $X$ . Let  $\Omega_X^1$  denote the sheaf of differential forms of degree 1 on  $X$ . Let  $\Omega_X^1(Y)$  be the smallest subsheaf of  $j_*\Omega_{X \setminus Y}^1$  that contains  $\Omega_X^1$  and  $d \log f$  for each holomorphic function  $f$  such that  $f^{-1}(0) \subset Y$ . Set  $\Omega_X^p(Y) = \wedge^p \Omega_X^1(Y)$ . The local sections of  $\Omega_X^p(Y)$  are called logarithmic differential forms with poles along  $Y$ .

Let  $Z$  be a smooth irreducible component of  $Y$ . We can associate to  $\alpha \in \Omega_X^1(Y)$  an holomorphic function  $\text{Res}_Z \alpha \in \mathcal{O}_Z$ , the *Poincaré residue of  $\alpha$  along  $Z$* . Assume that we are in the situation of (2.1),  $\alpha|_U = \sum_{i=1}^\nu \alpha_i dx_i / x_i + \sum_{i=\nu+1}^n \alpha_i dx_i$  and  $Z \cap U = \{x_j = 0\}$ , where  $1 \leq j \leq \nu$ . Then  $\text{Res}_Z \alpha|_{U \cap Z} = \alpha_j|_{U \cap Z}$ . Let  $W$  be the intersection of the smooth irreducible components  $Y_1, \dots, Y_\mu$  of  $Y$ . We call *residual submanifold of  $X$  along  $W$*  to the set of points  $o \in W$  such that  $\text{Res}_{Y_i}(\theta)$  vanishes at  $o$  for  $1 \leq i \leq \mu$ . We will denote the residual submanifold of  $X$  along  $W$  by  $R_W X$ .

A group action  $\alpha : \mathbb{C}^* \times X \rightarrow X$  on manifold  $X$  is called a free group action of  $\mathbb{C}^*$  on  $X$  if, for each  $x \in X$ , the isotropy subgroup  $\{t \in \mathbb{C}^* : \alpha(t, x) = x\}$  equals  $\{1\}$ . A manifold  $X$  with a free group action  $\alpha$  of  $\mathbb{C}^*$  is called a homogeneous manifold. We associate to each free group action  $\alpha$  of  $\mathbb{C}^*$  on  $X$  a vector field  $\rho$ , the *Euler vector field of  $\alpha$* , given by

$$\rho f = \frac{\partial}{\partial t} \alpha_x^* f|_{t=1}, f \in \mathcal{O}_X,$$

where  $\alpha_x(t) = \alpha(t, x)$ . Given homogeneous manifolds  $(X_1, \alpha_1)$  and  $(X_2, \alpha_2)$ , a holomorphic map  $\varphi : X_1 \rightarrow X_2$  is called *homogeneous* if  $\alpha_{2,t} \varphi = \varphi \alpha_{1,t}$ , for any  $t \in \mathbb{C}^*$ .

Let us recall some definitions and some results introduced in [9].

**Definition 2.1.** Let  $X$  be a manifold of dimension  $2n$  endowed with a free group action  $\alpha$ . Let  $Y$  be a divisor with normal crossings of  $X$ . If  $\sigma$  is a locally exact section of  $\Omega_X^2(Y)$ ,  $\alpha_t^* \sigma = t\sigma$  for each  $t \in \mathbb{C}^*$  and  $\sigma^n$  is a generator of  $\Omega_X^{2n}(Y)$  we say that  $\sigma$  is a *homogeneous logarithmic symplectic form with poles along  $Y$*  and  $(X, \sigma)$  a *homogeneous logarithmic symplectic manifold with poles along  $Y$* .

Let  $(X_1, \sigma_1), (X_2, \sigma_2)$  be homogeneous symplectic manifolds. A homogeneous map  $\varphi : X_1 \rightarrow X_2$  is a *homogeneous symplectic transformation* if  $\varphi^* \sigma_2 = \sigma_1$ .

If  $Y$  is the empty set we get the usual definition of homogeneous symplectic manifold.

Given a homogeneous logarithmic symplectic manifold  $(X, \sigma)$  we call  $\theta = \iota(\rho)\sigma$  the *canonical 1-form* of  $(X, \sigma)$ , where  $\iota(\rho)\sigma$  is the contraction of  $\rho$  and  $\sigma$ . Notice that  $\sigma = d\theta$ .

Given a vector bundle  $E$  over a manifold  $M$  we denote by  $\mathring{E}$  the manifold  $E \setminus M$ , where we identify  $M$  with the image of the zero section of  $E$ .

**Example 2.2.** Let  $M$  be a manifold and  $N$  a divisor with normal crossings of  $M$ . Let  $\pi : T^*(M/N) \rightarrow M$  be the vector bundle with sheaf of sections  $\Omega_M^1(N)$ . We will call  $T^*(M/N)$  the *logarithmic cotangent bundle of  $M$  along  $N$* .

The manifold  $\mathring{T}^*(M/N)$  has a canonical structure of logarithmic symplectic manifold with poles along  $\pi^{-1}(N)$ . Actually, there is a canonical section  $\theta$  of  $\Omega_{T^*(M/N)}^1(\pi^{-1}(N))$ , the *canonical 1-form* of  $T^*(M/N)$ . Given an integer  $\nu$  and a system of local coordinates  $(x_1, \dots, x_n)$  on an open set  $U$  of  $X$  verifying (2.1) there is one and only one

family of holomorphic functions  $\xi_i, 1 \leq i \leq n$ , defined on  $\pi^{-1}(U)$  such that  $\theta|_{\pi^{-1}(U)}$  equals

$$(2.2) \quad \sum_{i=1}^{\nu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^n \xi_i dx_i.$$

Moreover, the 2-form  $\sigma = d\theta$  is a homogeneous symplectic form with poles along  $\pi^{-1}(Y)$ .

A homogeneous logarithmic symplectic manifold is locally isomorphic to  $\mathring{T}^*(M/N)$  in the category of homogeneous symplectic manifolds:

**Theorem 2.3.** *Let  $\sigma$  be a homogeneous logarithmic symplectic form on a manifold  $X$  with poles along a divisor with normal crossings  $Y$ . Given  $o \in X$  let  $\nu$  be the number of irreducible components of  $Y$  at  $o$ . Then there is a system of local coordinates*

$$(2.3) \quad (x_1, \dots, x_n, \xi_1, \dots, \xi_n)$$

*on an open conic neighbourhood  $U$  of  $o$  such that  $Y \cap U = \{x_1 \cdots x_\nu = 0\}$ ,  $x_1, \dots, x_n$  are homogeneous of degree 0,  $\xi_1, \dots, \xi_n$  are homogeneous of degree 1 and  $\sigma|_U$  equals*

$$(2.4) \quad \sum_{i=1}^{\nu} d\xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^n d\xi_i dx_i.$$

A symplectic form  $\sigma$  on a manifold  $X$  defines on  $X$  a Poisson bracket  $\{\cdot, \cdot\}$ . Moreover, we can recover  $\sigma$  from the Poisson bracket. A submanifold  $W$  of  $X$  is called *involutive* [*invariant*] if  $\{I_W, I_W\} \subset I_W$  [ $\{I_W, \mathcal{O}_X\} \subset I_W$ ].

Given coordinates (2.3) on a conic neighbourhood  $U$  of  $o$  such that  $\sigma|_U$  equals (2.4), we have that  $\iota(\rho)\sigma|_U$  equals (2.2) and  $\{f, g\}$  equals

$$(2.5) \quad \sum_{i=1}^{\nu} x_i \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right) + \sum_{i=\nu+1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right)$$

for each  $f, g \in \mathcal{O}_X(U)$ .

**Proposition 2.4.** *Let  $X$  be an homogeneous logarithmic symplectic manifold with poles along a smooth divisor  $Y$ . Let  $W$  be the intersection of the smooth irreducible components  $Y_1, \dots, Y_\mu$  of  $Y$ . Then:*

- (i)  $X, R_W X$  are involutive submanifolds of  $X$ .
- (ii) The manifold  $R_W X$  has a canonical structure of homogeneous symplectic manifold with poles along the divisor induced in  $W$  by  $Y$ .

*Proof.* Let  $o \in W$ . There is a system of symplectic coordinates (2.3) on a conic open set  $U$  that contains  $o$  such that  $\theta|_U$  equals (2.2) and  $W \cap U = \{x_1 = \cdots = x_\mu = 0\}$ . Hence  $R_W X \cap U = \{x_1 = \cdots = x_\mu = \xi_1 = \cdots = \xi_\mu = 0\}$ . The restriction to  $R_W X \cap U$  of the Poisson bracket of  $X$  is given by

$$\{f, g\} = \sum_{i=\mu+1}^{\nu} x_i \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right) + \sum_{i=\nu+1}^n \left( \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} \right).$$

Hence  $R_W X \cap U$  is endowed with a 1-form  $\sum_{i=\mu+1}^{\nu} \xi_i dx_i / x_i + \sum_{i=\nu+1}^n \xi_i dx_i$ .  $\square$

**Definition 2.5.** Let  $X$  be a manifold of dimension  $2n + 1$  and  $Y$  a divisor with normal crossings of  $X$ . A local section  $\omega$  of  $\Omega_X^1\langle Y \rangle$  is called a *logarithmic contact form with poles along  $Y$*  if  $\omega(dw)^n$  is a local generator of  $\Omega_X^{2n+1}\langle Y \rangle$ .

We say that a locally free sub  $\mathcal{O}_X$ -module  $\mathcal{L}$  of  $\Omega_X^1\langle Y \rangle$  is a *logarithmic contact structure on  $X$  with poles along  $Y$*  if it is locally generated by a logarithmic contact forms with poles along  $Y$ . We say that a manifold with a logarithmic contact structure with poles along a divisor with normal crossings  $Y$  is a *logarithmic contact manifold with poles along  $Y$* . We call  $Y$  the *set of poles of the logarithmic contact manifold  $(X, \mathcal{L})$* .

Let  $(X_1, \mathcal{L}_1), (X_2, \mathcal{L}_2)$  be logarithmic contact manifolds. We say that a holomorphic map  $\varphi : X_1 \rightarrow X_2$  is a *contact transformation* if  $\varphi^*\omega$  is a local generator of  $\mathcal{L}_1$  for each local generator  $\omega$  of  $\mathcal{L}_2$ .

Let  $X$  be a homogeneous logarithmic symplectic manifold. Let  $\theta$  be the canonical 1-form of  $X$  and let  $Y$  be the set of poles of  $X$ . Let  $X_*$  be the quotient of  $X$  by its  $\mathbb{C}^*$  action. Then  $X_*$  is a manifold and the canonical epimorphism  $\gamma : X \rightarrow X_*$  is a  $\mathbb{C}^*$ -bundle. Put  $Y_* = \gamma(Y)$ . Let  $\mathcal{L}$  be the sub  $\mathcal{O}_{X_*}$ -module of  $\Omega_{X_*}^1\langle Y \rangle$  generated by the logarithmic differential forms  $s^*\theta$ , where  $s$  is a holomorphic section of  $\gamma$ . Then  $\mathcal{L}_*$  is a structure of logarithmic contact manifold with poles along  $Y_*$ .

We constructed on this way a functor *contactification*  $X \mapsto X_*$  from the category  $\mathcal{S}$  of homogeneous logarithmic symplectic manifolds into the category  $\mathcal{C}$  of logarithmic contact manifolds. Let  $\mathcal{S}' [\mathcal{C}']$  be the subcategory of  $\mathcal{S} [\mathcal{C}]$  such that its morphisms are locally injective. The functor contactification defines an equivalence of categories from  $\mathcal{S}'$  onto  $\mathcal{C}'$ .

Theorem 4.3 constructs an example of a morphism of homogeneous logarithmic symplectic manifolds that is not locally injective. Theorem 4.4 constructs an example of a morphism of logarithmic contact manifolds that is not locally injective. Moreover, this last morphism does not come from a morphism of homogeneous logarithmic symplectic manifolds through the functor  $X \mapsto X_*$ . This is the main reason why there is no equivalence of categories between  $\mathcal{S}$  and  $\mathcal{C}$  and we have to use logarithmic contact manifolds in this paper. This phenomena has no equivalent in the classic, non logarithmic case.

It is common to use the coordinates of  $\mathbb{C}^n$  when dealing with the projective space  $\mathbb{CP}^{n-1}$ . We will use symplectic coordinates when dealing with logarithmic contact manifolds within the same spirit. In particular we do not feel the need to define concepts like involutivity or residual set in the contact context.

The *projective logarithmic cotangent bundle of  $M$  with poles along  $N$*   $\mathbb{P}^*\langle M/N \rangle = (\mathring{T}^*\langle M/N \rangle)_*$  has a canonical structure of logarithmic contact manifold.

### 3. LEGENDRIAN VARIETIES

Let  $(X, \mathcal{L})$  be a contact manifold of dimension  $2n + 1$ . An analytic subset  $\Gamma$  of  $X$  is a Legendrian variety of  $X$  if it verifies the following three conditions:  $\Gamma$  has dimension  $n$ ,  $\Gamma$  is involutive and the restriction to the regular part of  $\Gamma$  of a local generator of  $\mathcal{L}$  vanishes.

Each two of these three conditions imply the remaining one.

Given a manifold  $M$  and an irreducible analytic subset  $S$  of  $M$  there is one and only one Legendrian variety  $\mathbb{P}_S^*M$  of  $\mathbb{P}^*M$  such that  $\pi(\mathbb{P}_S^*M) = S$ . The analytic set  $\mathbb{P}_S^*M$  is called the *conormal* of  $S$  (see for instance [5]). If  $S$  has irreducible components  $S_i, i \in I$ , the conormal  $\mathbb{P}_S^*M$  of  $S$  equals  $\cup_{i \in I} \mathbb{P}_{S_i}^*M$ .

Let us introduce stratified versions of the definitions above.

**Definition 3.1.** Let  $X$  be a logarithmic contact manifold of dimension  $2n + 1$  with set of poles  $Y$ . An analytic subset  $\Gamma$  of  $X$  is called a Legendrian variety of  $X$  if the following conditions are verified:

- (i)  $\Gamma$  is involutive and  $\Gamma \setminus Y$  is a Legendrian variety of  $X \setminus Y$ .
- (ii) Let  $o$  be a point of  $\Gamma$ . Let  $\Gamma' [Y']$  be an irreducible component of the germ of  $\Gamma [Y]$  at  $o$ . If  $\Gamma' \subset Y'$ ,  $\Gamma' \subset R_{Y'}X$ .
- (iii) If  $\Gamma''$  is an irreducible component of  $\Gamma \cap Y'$  and  $\Gamma'' \not\subset Y_{\text{sing}}$ ,  $\Gamma''$  is a Legendrian subvariety of  $R_{Y'}X$ .

Let  $N$  be a divisor with normal crossings of a manifold  $M$ .

**Remark 3.2.** Let  $\Gamma$  be a Legendrian variety of  $\mathbb{P}^*\langle M/N \rangle$ . Let  $Q$  be an irreducible component of  $N$ . Let  $R$  be the divisor with normal crossings induced in  $Q$  by  $N$ . If  $\Gamma$  is contained in  $\pi^{-1}(Q)$ , it follows from condition (ii) of Definition 3.1 that  $\Gamma$  is contained in  $\mathbb{P}^*\langle Q/R \rangle$ .

An analytic subset  $S$  of  $M$  is a *natural analytic subset* of  $(M, N)$  if no germ of  $S$  is an intersection of irreducible components of a germ of  $N$ .

A Legendrian variety of a logarithmic contact manifold  $X$  with poles along  $Y$  is a natural analytic subset of  $(X, Y)$ .

**Definition 3.3.** Let  $S$  be a natural irreducible subset of  $(M, N)$ . Let  $Q$  be the intersection of the irreducible components of  $N$  that contain  $S$ . Let  $R$  be the divisor with normal crossings induced in  $Q$  by  $N$ . We call *conormal* of  $S$  (relative to  $N$ ) to the closure  $\mathbb{P}_S^*\langle Q/R \rangle$  of the conormal of the analytic subset  $S \setminus R$  of  $Q \setminus R$  in  $\mathbb{P}^*\langle Q/R \rangle$ .

Let  $S$  be a natural analytic subset of  $(M, N)$ . We call *conormal* of  $S$  to the union  $\mathbb{P}_S^*\langle M/N \rangle$  of the conormals of its irreducible components.

**Example 3.4.** Set  $M = \mathbb{C}^4$ ,  $N = \{x_1x_2x_3 = 0\}$ ,  $S = \{x_1 = x_2 = x_4 = 0\}$ . Hence  $Q = \{x_1 = x_2 = 0\}$ ,  $R = \{x_1 = x_2 = x_3 = 0\}$  and  $\mathbb{P}^*\langle Q/R \rangle = \{x_1 = x_2 = \xi_1 = \xi_2 = 0\}$  is endowed with the canonical 1-form  $\xi_3dx_3/x_3 + \xi_4dx_4$ . Therefore  $\mathbb{P}_S^*\langle M/N \rangle = \{x_1 = x_2 = \xi_1 = \xi_2 = x_4 = \xi_3 = 0\}$ .

**Theorem 3.5.** *The conormal of a natural analytic set is a Legendrian variety.*

*Proof.* Let  $S$  be a germ of a natural analytic subset of  $(M, N)$ . Set  $\Gamma = \mathbb{P}_S^*\langle M/N \rangle$ . We can assume that  $S$  is irreducible and that  $M$  is the intersection of the irreducible components of  $N$  that contain  $S$ . The intersection of  $\Gamma$  with  $\pi^{-1}(M \setminus N)$  is the Legendrian variety  $\mathbb{P}_{S \setminus N}^*(M \setminus N)$  of the contact manifold  $\mathbb{P}^*(M \setminus N)$ . Since  $\Gamma$  is the closure of  $\mathbb{P}_{S \setminus N}^*(M \setminus N)$ ,  $\Gamma$  is involutive. Hence condition (i) is verified. Condition (ii) follows from the definition of conormal variety.

Let us prove condition (iii) by induction in the dimension of  $M$ . Condition (iii) is trivial if  $\dim M = 1$ . Let  $Z$  be an irreducible component of  $\pi^{-1}(N)$ . The set  $Q = \pi(Z)$  is an irreducible component of  $N$ . Since  $\pi^{-1}(N)$  is invariant,  $Z$  is invariant. Let  $R$  be the divisor induced in  $Q$  by  $N$ . Let  $\Gamma_0$  be an irreducible component of  $\Gamma \cap Z$  that is not contained in the singular locus of  $\pi^{-1}(N)$ . Let us show that

$$(3.1) \quad \Gamma_0 \subset \mathbb{P}^*\langle Q/R \rangle.$$

It is enough to show that  $\gamma^{-1}(\Gamma_0)$  is contained in  $T^*\langle Q/R \rangle$ . Let  $o \in \gamma^{-1}(\Gamma_0 \setminus \pi^{-1}(N)_{\text{sing}})$ . There is an open conic neighborhood  $U$  of  $o$  and a system of local coordinates (2.3) on  $U$  such that

$$\theta|_U = \xi_1 \frac{dx_1}{x_1} + \sum_{i=2}^n \xi_i dx_i$$

and  $\gamma^{-1}(Z) \cap U = \{x_1 = 0\}$ . There is a holomorphic map  $\delta : \{t \in \mathbb{C} : |t| < 1\} \rightarrow \gamma^{-1}(\Gamma)$  such that

$$\gamma(\delta(0)) = o \text{ and } \delta^{-1}(\gamma^{-1}(\pi^{-1}(N))) = \{0\}.$$

Set  $\delta_i = x_i \circ \delta$ ,  $1 \leq i \leq n$ . Since  $\theta$  vanishes on  $\gamma^{-1}(\Gamma_0 \setminus Z)$ ,

$$(3.2) \quad \xi_1(\delta(t)) \frac{\delta'_1(t)}{\delta_1(t)} + \sum_{i=2}^n \xi_i(\delta(t)) \delta'_i(t) = 0 \quad \text{if } t \neq 0.$$

Hence

$$(3.3) \quad \xi_1(o) = 0,$$

and (3.1) holds.

Since  $Z$  is invariant,  $\Gamma \cap Z$  is an involutive submanifold of  $P^*\langle M/N \rangle$ . Hence  $\Gamma \cap Z$  is an involutive submanifold of  $\mathbb{P}^*\langle Q/R \rangle$ . Hence its irreducible components are involutive. Since  $\dim \Gamma_0 = \dim \Gamma - 1$ ,  $\Gamma_0 \setminus \pi^{-1}(R)$  is a Legendrian subvariety of  $\mathbb{P}^*\langle Q \setminus R \rangle$ . Let  $S_0$  be the closure in  $Q$  of the projection of  $\Gamma_0 \setminus \pi^{-1}(R)$ . Then  $\Gamma_0$  is the conormal of  $S_0$ . By the induction hypothesis,  $\Gamma_0$  is a Legendrian variety of  $\mathbb{P}^*\langle Q/R \rangle$ .  $\square$

**Theorem 3.6.** *An irreducible Legendrian subvariety of a projective logarithmic cotangent bundle is the conormal of its projection.*

*Proof.* The result is known for Legendrian subvarieties of a projective cotangent bundle (see for instance [11]). The theorem is an immediate consequence of this particular case.  $\square$

#### 4. BLOW UPS

Let  $D$  be a submanifold of  $M$ . The vector bundle  $T_DM$  defined by the exact sequence of vector bundles  $0 \rightarrow TD \rightarrow D \times_M TM \rightarrow T_DM \rightarrow 0$  is called the *normal bundle* of  $M$  along  $D$ .

**Lemma 4.1.** *Let  $f : X \rightarrow Y$  be a holomorphic map between manifolds. Let  $A \subset [B]$  be a submanifold of  $X$  [Y]. If  $f(A) = B$  and  $f|_A : A \rightarrow B$  are submersions,  $f$  induces a canonical holomorphic map  $\sigma$  from  $T_AX$  into  $T_BY$ .*

*Proof.* Given  $o \in X$ ,  $Df(o)$  defines maps from  $T_oX$  onto  $T_{f(o)}Y$  and from  $T_oA$  onto  $T_{f(o)}B$ . Hence  $Df(o)$  induces a map from  $T_oX/T_oA$  onto  $T_{f(o)}X/T_{f(o)}B$ . Therefore  $Df$  induces a map  $\sigma : T_AX \rightarrow T_BY$ . Locally there are coordinates  $(x_1, \dots, x_a, y_1, \dots, y_b, z_1, \dots, z_c, w_1, \dots, w_d)$  on  $X$  and  $(u_1, \dots, u_a, v_1, \dots, v_c)$  on  $Y$  such that  $A = \{z = w = 0\}$ ,  $B = \{v = 0\}$  and  $f(x, y, z, w) = (x, z)$ . Hence there are local coordinates

$$(x_1, \dots, x_a, y_1, \dots, y_b, \tilde{z}_1, \dots, \tilde{z}_c, \tilde{w}_1, \dots, \tilde{w}_d)$$

on  $T_A X$  and  $(u_1, \dots, u_a, \tilde{v}_1, \dots, \tilde{v}_c)$  on  $T_B Y$  such that  $A = \{\tilde{z} = \tilde{w} = 0\}$ ,  $B = \{\tilde{v} = 0\}$  and  $\sigma(x, y, \tilde{z}, \tilde{w}) = (x, \tilde{z})$ .  $\square$

Let  $X_M^\#$  be the normal deformation of  $M$  in  $X$  (see section 4.1 of [5]). Let  $\tau : \tilde{X}_M \rightarrow X$  be the blow up of  $X$  along  $M$ . Set  $E = \tau^{-1}(M)$ . There are maps  $\Phi : X_M^\# \rightarrow \tilde{X}_M$ ,  $p : X_M^\# \rightarrow X$  and  $s : X_M^\# \rightarrow \mathbb{C}^*$  such that:

- (i)  $p^{-1}(X \setminus M) \simeq (X \setminus M) \times \mathbb{C}^*$ ;
- (ii)  $s^{-1}\mathbb{C}^* \simeq X \times \mathbb{C}^*$ ,  $s^{-1}(\{0\}) \simeq T_M X$ ;
- (iii)  $p = \tau\Phi$ .

There is a free action of  $\mathbb{C}^*$  on  $X_M^\#$  such that  $\Phi$  induces an isomorphism from  $X_M^\#/\mathbb{C}^*$  into  $\tilde{X}_M$  that takes  $\mathbb{P}_M X$  into  $E$ .

Let  $\tilde{S} [\tilde{C}_D(S)]$  be the proper inverse image of  $S$  by  $\tau [p]$ . The set  $C_D(S) = \tilde{C}_D(S) \cap T_M X$  is called the *normal cone of  $S$  along  $M$* . We recall that  $\Phi(C_D(S)) = \tilde{S} \cap E$ . Set  $X = \mathbb{C}^{a+b+c}$  with coordinates  $(x_1, \dots, x_a, y_1, \dots, y_b, z_1, \dots, z_c)$ .

Set  $\Lambda = \{x = y = 0\}$  and set  $L = \{(\tilde{x}, \tilde{y}, z) \in T_\Lambda X : \tilde{x} = 0\}$ . The blow up of  $X$  along  $\Lambda$  is the glueing of the affine open sets  $U_{x_i}$ ,  $1 \leq i \leq a$ ,  $U_{y_j}$ ,  $1 \leq j \leq b$ . Here  $U_{x_i}$  is the affine set with coordinates  $(\frac{x_1}{x_i}, \dots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \dots, \frac{x_a}{x_i}, \frac{y_1}{x_i}, \dots, \frac{y_b}{x_i}, z_1, \dots, z_c)$ .

**Lemma 4.2.** *Let  $\Gamma$  be the germ of a closed analytic subset of  $X$ .*

*If  $C_\Lambda(\Gamma) \cap L \subset \{\tilde{x} = \tilde{y} = 0\}$ ,  $\tilde{\Gamma} \cap E \subset \cup_{i=1}^a U_{x_i}$ .*

*Proof.* Notice that  $\mathbb{C}_\Lambda^{a+b+c\#} = \{(s, \tilde{x}, \tilde{y}, z) : (\tilde{x}, \tilde{y}) \neq (0, 0)\}$ . Moreover,  $\Phi$  induces a surjective map

$$(s, \tilde{x}, \tilde{y}, z) \mapsto \left( \frac{\tilde{x}_1}{\tilde{x}_i}, \dots, s\tilde{x}_i, \dots, \frac{\tilde{x}_a}{\tilde{x}_i}, \frac{\tilde{y}_1}{\tilde{x}_i}, \dots, \frac{\tilde{y}_b}{\tilde{x}_i}, z_1, \dots, z_c \right)$$

from  $\{(s, \tilde{x}, \tilde{y}, z) : \tilde{x}_i \neq 0\}$  into  $U_{x_i}$ . Hence  $E \cap U_{x_i} = \Phi(\{s = 0, \tilde{x}_i \neq 0\})$  and  $E \setminus U_{x_i} = \Phi(\{s = 0, \tilde{x}_i = 0\})$ . Since  $E \setminus \cup_{i=1}^a U_{x_i} = \Phi(\{s = 0, \tilde{x} = 0\})$ ,

$$(\tilde{\Gamma} \cap E) \setminus \cup_{i=1}^a U_{x_i} = \Phi(C_\Lambda(\Gamma) \cap L).$$

$\square$

Let  $L$  be a submanifold of a manifold  $M$  of codimension greater or equal than 2. Let  $N$  be a normal crossings divisor of  $M$ . We say that  $L$  is a *center* of  $(M, N)$  at  $o \in L \cap N$  if there are manifolds  $\Sigma_1, \dots, \Sigma_s$  and  $I, J \subset \{1, \dots, s\}$  such that  $\Sigma_1 \cup \dots \cup \Sigma_s$  is a divisor with normal crossings at  $o$  and

$$(N, o) = (\cup_{i \in I} \Sigma_i, o), (L, o) = (\cap_{i \in J} \Sigma_i, o).$$

We say that  $L$  is a *trivial center* [non-trivial center] of  $(M, N)$  at  $o$ , if  $J \subset I$  [ $J \not\subset I$ ].

**Theorem 4.3.** *Let  $M$  be a manifold and let  $N$  be a divisor with normal crossings of  $M$ . Let  $L$  be a trivial center of  $(M, N)$ . Let  $\rho : \tilde{M} \rightarrow M$  be the blow up of  $M$  along  $L$ . Set  $\tilde{N} = \rho^{-1}(N)$ .*

- (i) *The blow up of  $T^*\langle M/N \rangle$  along  $\pi^{-1}(L)$  is a logarithmic symplectic manifold isomorphic to  $T^*\langle \tilde{M}/\tilde{N} \rangle$  and diagram (4.1) commutes.*

$$(4.1) \quad \begin{array}{ccc} T^*\langle M/N \rangle & \leftarrow & T^*\langle \tilde{M}/\tilde{N} \rangle \\ \downarrow & & \downarrow \\ M & \leftarrow & \tilde{M} \end{array}$$

- (ii) *If  $S$  is a natural hypersurface of  $M$ , the proper inverse image of the conormal of  $S$  equals the conormal of the proper inverse image of  $S$ .*

*Proof.* The proof of statement (i) is similar to the proof of statement (ii) of Theorem 4.4. Hence we omit it. Assume  $S$  irreducible. Set  $\Gamma = T_S^*\langle M \setminus N \rangle$ . Let  $\tilde{\Gamma}$  be the proper inverse image of  $\Gamma$  by the blow up with center  $\pi^{-1}(L)$ . Since diagram (4.1) commutes, the projection of  $\tilde{\Gamma}$  into  $\tilde{M}$  equals the proper inverse image  $\tilde{S}$  of  $S$  by  $\rho$ . Since  $\tilde{\Gamma} \setminus \pi^{-1}(\tilde{N})$  is the conormal of  $\tilde{S} \setminus \tilde{N}$ ,  $\tilde{\Gamma} = T_{\tilde{S}}^*\langle \tilde{M} \setminus \tilde{N} \rangle$ .  $\square$

Let  $X$  be a manifold and let  $Y$  be a closed hypersurface of  $X$ . We will denote by  $\mathcal{O}_X(Y)$  the sheaf of meromorphic functions  $f$  such that  $fI_Y \subset \mathcal{O}_X$ .

**Theorem 4.4.** *Let  $N$  be the normal crossings divisor of a manifold  $M$ . Let  $L$  be a nontrivial center of  $(M, N)$ . Let  $\tau$  be the blow up of  $X = \mathbb{P}^*\langle M/N \rangle$  along  $\Lambda = \mathbb{P}_L^*\langle M/N \rangle$ . Set  $E = \tau^{-1}(\Lambda)$ . Let  $\rho : \tilde{M} \rightarrow M$  be the blow up of  $M$  along  $L$ . Set  $\tilde{N} = \rho^{-1}(N)$ .*

- (i) *If  $\mathcal{L}$  is the canonical contact structure of  $\mathbb{P}^*\langle M/N \rangle$ , the  $\mathcal{O}_{\tilde{X}}$ -module  $\mathcal{O}_{\tilde{X}}(E)\tau^*\mathcal{L}$  is a structure of logarithmic contact manifold on  $\tilde{X}$  with poles along  $\tau^{-1}(\pi^{-1}(M))$ .*
- (ii) *There is an injective contact transformation  $\varphi$  from a dense open subset  $\Omega$  of  $\tilde{X}$  onto  $P^*\langle \tilde{M}/\tilde{N} \rangle$  such that diagram (4.2) commutes.*

$$(4.2) \quad \begin{array}{ccccc} P^*\langle M/N \rangle & \xleftarrow{\tau} & \tilde{X} \hookleftarrow \Omega & \xrightarrow{\varphi} & P^*\langle \tilde{M}/\tilde{N} \rangle \\ \pi \downarrow & & & & \downarrow \pi \\ M & & \xleftarrow{\rho} & & \tilde{M} \end{array}$$

- (iii) *Let  $S$  be a germ of a natural analytic subset of  $(M, N)$  at  $o \in N$ . Set  $\Gamma = \mathbb{P}_S^*\langle M/N \rangle$ . Let  $\tilde{S}$  be the proper inverse image of the blow up of  $M$  along  $L$ . If  $S$  has trivial limits of tangents at  $o$  and  $C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \Lambda$ , then  $\tilde{\Gamma} \subset \Omega$  and  $\varphi(\tilde{\Gamma}) = \mathbb{P}_{\tilde{S}}^*\langle \tilde{M}/\tilde{N} \rangle$ , where  $\sigma$  denotes the canonical projection from  $T_\Lambda \mathbb{P}^*\langle M/N \rangle$  onto  $T_L M$  introduced in Lemma 4.1.*

*Proof.* The proof of statement (i) is quite similar to the proof of Proposition 9.2 of [9]. Hence we omit it.

- (ii) Assume  $M = \mathbb{C}^{n+1}$ ,  $N = \{x_1 \cdots x_\nu = 0\}$  and  $X = \mathbb{P}^*\langle M/N \rangle$ .

The canonical 1-form  $\theta$  of  $T^*\langle M/N \rangle$  equals

$$\sum_{i=1}^{\nu} \xi_i \frac{dx_i}{x_i} + \sum_{i=\nu+1}^{n+1} \xi_i dx_i.$$

Hence there is  $\iota$  such that  $L = \{x_\iota = \cdots = x_k = x_{n+1} = 0\}$ . Let  $\hat{\tilde{X}}$  be the homogeneous symplectic manifold associated to  $\tilde{X}$ . Let  $\hat{\theta}$  be the canonical 1-form of  $\hat{\tilde{X}}$ . By the argument of (i)  $\hat{\tilde{X}}$  is the union of open set  $\hat{U}_j, j = \iota, \dots, k, n+1$  and  $\hat{V}_j, j = 1, \dots, \nu, k+1, \dots, n$ .

Set  $\hat{\Omega} = \cup_j \hat{U}_j$ . Set  $\Omega = \hat{\Omega}_*, \hat{\theta}_j = \hat{\theta}|_{\hat{U}_j}, j = 2, \dots, k, n+1$ . Endow  $\mathbb{C}^{2n+2}$  with the coordinates

$$x_1, \dots, x_{\iota-1}, \frac{x_\iota}{x_j}, \dots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \dots, \frac{x_\nu}{x_j}, x_{\nu+1}, \dots, x_n, \frac{x_{n+1}}{x_j}, \eta_1, \dots, \eta_{n+1}.$$

We can assume that  $\hat{U}_j = \{(\eta_1, \dots, \eta_{n+1}) \neq (0, \dots, 0)\}$  and

$$\hat{\theta}_j = \sum_{i=1}^{\iota-1} \eta_i \frac{dx_i}{x_i} + \sum_{i=\iota}^{\nu} \eta_i \frac{d \frac{x_i}{x_j}}{\frac{x_i}{x_j}} + \eta_j dx_j + \sum_{i=\nu+1}^k \eta_i d \frac{x_i}{x_j} + \sum_{i=k+1}^n \eta_i dx_i + \eta_{n+1} d \frac{x_{n+1}}{x_j}.$$



The blow up of  $M$  along  $L$  is the glueing of the open affine sets  $M_j, j = \iota, \dots, k, n+1$  where  $M_j$  is associated to the generator  $x_j$  of the defining ideal  $L$ . Hence  $T^*\langle \widetilde{M}/\widetilde{N} \rangle$  is the glueing of the open affine sets  $T^*\langle M_j/\widetilde{N} \cap M_j \rangle, j = \iota, \dots, k, n+1$ . Set  $\widehat{W}_j = \mathring{T}^*\langle M_j/N \cap M_j \rangle$ . Let  $\widetilde{\theta}$  be the canonical 1-form of  $T^*\langle M/N \rangle$ . Set  $\widetilde{\theta}_j = \widetilde{\theta}|_{\widehat{W}_j}$ . Endow  $\mathbb{C}^{2n+2}$  with the coordinates

$$x_1, \dots, x_{\iota-1}, \frac{x_{\iota}}{x_j}, \dots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \dots, \frac{x_{\nu}}{x_j}, x_{\nu+1}, \dots, x_n, \frac{x_{n+1}}{x_j}, \zeta_1, \dots, \zeta_{n+1}.$$

We can assume that  $\widehat{W}_j = \{(\zeta_1, \dots, \zeta_{n+1}) \neq (0, \dots, 0)\}$  and

$$\widetilde{\theta}_j = \sum_{i=1}^{\iota-1} \zeta_i \frac{dx_i}{x_i} + \sum_{\substack{i=\iota \\ i \neq j}}^{\nu} \zeta_i \frac{d\frac{x_i}{x_j}}{\frac{x_i}{x_j}} + \zeta_j \frac{dx_j}{x_j} + \sum_{\substack{i=\nu+1 \\ i \neq j}}^k \zeta_i d\frac{x_i}{x_j} + \sum_{i=k+1}^n \zeta_i dx_i + \zeta_{n+1} d\frac{x_{n+1}}{x_j}.$$

Since

$$\widehat{\widetilde{X}} \hookleftarrow T^*\langle M \setminus L/N \setminus L \rangle \simeq T^*\langle \widetilde{M} \setminus \rho^{-1}(L)/\widetilde{N} \setminus \rho^{-1}(L) \rangle \hookrightarrow T^*\langle \widetilde{M} \setminus \widetilde{N} \rangle$$

There is a bimeromorphic contact transformation  $\widehat{\varphi}^{-1} : \widehat{\widetilde{X}} \rightarrow \mathring{T}^*\langle \widetilde{M}/\widetilde{N} \rangle$ . It is enough to show that the domain of  $\widehat{\varphi}$  contains  $\widehat{\Omega}$  and its image equals  $\mathring{T}^*\langle \widetilde{M}/\widetilde{N} \rangle$ . Since

$$\widehat{U}_j \setminus \tau^{-1}(\pi^{-1}(L)) = \widehat{W}_j \setminus \pi^{-1}(\rho^{-1}(L)),$$

$\eta_i = \zeta_i$  on a dense open set of their domain. Hence  $\eta_i = \zeta_i$  everywhere and the domain of  $\widehat{\varphi}$  contains  $U_j$  for  $j = \iota, \dots, k, n+1$ .

(iii) The statement follows from Lemma 4.2 and the arguments of the proof of statement (ii) of theorem 4.3.  $\square$

## 5. MAIN RESULT

Let  $N$  be a germ of normal crossings divisor of a three dimensional manifold  $M$  at a point  $o$ . Let  $S$  be a germ of a natural irreducible surface of  $M$  at  $o$ . We say that  $S$  is *quasi-ordinary* at  $o$  if there is a system of local coordinates  $(x, y, z)$  on a neighbourhood of  $o$  such that the discriminant of  $S$  relatively to the projection  $(x, y, z) \mapsto (x, y)$  is contained in  $\{xy = 0\}$ . There is a positive integer  $m$  and  $\zeta \in \mathbb{C}\{x, y\}$  such that  $z = \zeta(x^{\frac{1}{m}}, y^{\frac{1}{m}})$  defines a parametrization of  $S$ . If  $\zeta \neq 0$ , let  $x^\lambda y^\mu$  be the monomial of lowest degree that occurs in  $\zeta$ . If  $\lambda\mu = 0$  or  $\lambda, \mu \in \mathbb{Z}$  and there is a monomial of  $\zeta$  that does not verify this condition, let  $x^a y^b$  be the monomial of lowest degree of  $\zeta$  that does not verify this condition. Otherwise, set  $a = b = +\infty$ . If  $\zeta = 0$ , we set also  $\lambda = \mu = +\infty$ . We call  $\lambda, \mu, a, b$ , the *exponents* of  $\zeta$ .

We say that  $\zeta$  is in *strong normal form* if  $\zeta$  has no monomials with integer exponents and

$$(5.1) \quad \lambda > \mu \text{ or } \lambda = \mu \text{ and } a \geq b.$$

We say that  $\zeta$  is in *normal form* if  $\zeta = 0$  or (5.1) holds.

Lipman proved a desingularization theorem for quasi-ordinary surfaces (see [7]). It was not clear that Lipman's result produced an embedded desingularization because the parametrization is related to a system of local coordinates and the choice of the centers is dependent on another system of local coordinates, related to the divisors created by successive explosions. Ban and McEwan showed that Lipman's algorithm produces an embedded desingularization that coincides with the general algorithm proposed by Bierstone and Milman [3]. Theorem 2.5 of [2] shows that after all the two systems of local coordinates are not that different. This result is a key point of their proof. Theorem 5.2 is a slightly more precise version of theorem 2.5 of [2]. We will need it in order to prove Theorem 5.3.

**Definition 5.1.** Let  $M$  be a manifold of dimension 3. Let  $N$  be a normal crossings divisor of  $M$ . Let  $S$  be a natural surface of  $(M, N)$ . Let  $o \in S \cap N$ . We say that  $N$  is *adapted to  $S$  at  $o$*  if there is a system of local coordinates  $(x, y, z)$  centered at  $o$  such that  $(N, o) \subset \{xyz = 0\}$ , the discriminant of the germ of  $S$  relatively to the projection  $(x, y, z) \mapsto (x, y)$  is contained in  $\{xy = 0\}$  and the parametrization  $z = \varphi(x^{\frac{1}{m}}, y^{\frac{1}{m}})$  of  $(S, o)$  is in normal form. We say that  $N$  is *adapted to  $S$*  if  $N$  is adapted to  $S$  at  $o$ , for each  $o \in S \cap N$ .

We will denote by  $\varepsilon$  or  $\varepsilon_i$  a unit of  $\mathbb{C}\{x^{\frac{1}{m}}, y^{\frac{1}{m}}\}$  and by  $\delta$  or  $\delta_i$  a unit of  $\mathbb{C}\{x^{\frac{1}{m}}\}$ , for a convenient  $m$ .

**Theorem 5.2.** *Let  $S_0$  be the germ of a quasi-ordinary surface. At each step of Lipman's algorithm the normal crossings divisor we obtain is adapted to the proper inverse image of  $S$ .*

*Proof.* We prove the theorem by induction on the number of steps. Let  $S$  be the proper inverse image of  $S_0$  at some step of Lipman's algorithm and  $N$  the system of exceptional divisor at that step. Let  $o \in S \cap N$ . We assume that  $S$  is singular at  $o$ . Let  $(x, y, z)$  a system of coordinates centered at  $o$  such that  $S$  is defined at  $o$  by the parametrization  $z = x^\lambda y^\mu \varepsilon$  and  $(N, o) \subset \{xyz = 0\}$ . For this proof we will

only consider the following types of changes of coordinates:

$$(5.2) \quad (x, y, z) \mapsto (y, x, z),$$

$$(5.3) \quad (x, y, z) \mapsto (z, y, x),$$

$$(5.4) \quad (x, y, z) \mapsto (z, x, y),$$

$$(5.5) \quad (x, y, z) \mapsto (x, y, z - q), \quad q \in \mathbb{C}\{x, y\}.$$

We will use the notation of table 1 for the center of the resolution step.

Assume  $\lambda + \mu < 1$ . Then we blow up  $\sigma_0$ . Let  $\tilde{S}$ ,  $\tilde{N}$  be the proper inverse images of, respectively,  $S$  and  $N$ . Set  $U_x$  as the open affine chart defined by the coordinates  $(x, \frac{y}{x}, \frac{z}{x})$ . Then, after a re-parametrization,

$$(5.6) \quad \frac{y}{x} = \left(\frac{z}{x}\right)^{\frac{1}{\mu}} x^{\frac{1-(\lambda+\mu)}{\mu}} \varepsilon_1$$

and  $\tilde{N} \cap U_x \subset \{x \frac{y}{x} \frac{z}{x} = 0\}$ . Notice that  $\frac{1}{\mu} > \frac{1-(\lambda+\mu)}{\mu}$  and  $\frac{1}{\mu} > 1$ . Furthermore, if  $(\lambda, \mu) = (a/n, 1/n)$ , for some positive integers  $a, n$ ,  $\frac{1}{\mu}$  and  $\frac{1-(\lambda+\mu)}{\mu}$  are positive integers. After a change of coordinates of type (5.4),  $\tilde{N}$  is adapted to  $\tilde{S}$  at the origin of  $U_x$ . From (5.6), one can easily check that, after a appropriate change of coordinates,  $\tilde{N}$  is adapted to  $\tilde{S}$  at the points of  $U_x$  of the type  $(0, 0, a)$  and  $(a, 0, 0)$ ,  $a \in \mathbb{C}^*$ . On the open affine chart defined by the coordinates  $(\frac{x}{y}, y, \frac{z}{y})$  the reasoning is analogous. On the open affine chart defined by the coordinates  $(\frac{x}{z}, \frac{y}{z}, z)$ , no re-parametrization is needed and one can easily check that  $\tilde{N}$  is adapted to  $\tilde{N}$  on this chart. Notice that if on this chart the multiplicity drops, we return to the case  $\lambda + \mu \geq 1$ .

Assume  $\lambda + \mu \geq 1$ . Let  $p \in \mathbb{C}\{x, y\}$  such that  $z - p$  has no monomials with integer exponents. Assume that for  $z - p$ ,  $1 < \lambda < 2$  and  $\mu = 0$ . Then we blow up  $\sigma_x$ . On the open affine set  $V_x$  defined by the coordinates  $(x, y, \frac{z}{x})$ , possibly after a change of coordinates of the type (5.5) with  $q \in \mathbb{C}^*$ , we need to do a re-parametrization and we obtain

$$x = \left(\frac{z}{x}\right)^{\frac{1}{1-\lambda}} \varepsilon_1$$

and  $\tilde{N} \cap U_x \subset \{xy \frac{z}{x} = 0\}$ . Notice that if  $\lambda = \frac{n-1}{n}$ , for some positive integer  $n$ ,  $\frac{1}{1-\lambda}$  is a positive integer. From this parametrization one can easily check that  $\tilde{N}$  is adapted to  $\tilde{S}$ . The remaining cases need no re-parametrization and one can easily check that the theorem holds.

We remark that if  $\lambda, \mu \in \mathbb{Z}$ , at some step of Lipman's algorithm, before we reach the situation where  $\lambda + \mu < 1$ , we do succession of resolution steps with center  $\sigma_x$  or  $\sigma_y$ . Eventually we reach a situation where we can do a change of coordinates of type (5.5) with  $q = a + p$ ,  $a \in \mathbb{C}^*$ . Hence, for  $o \in S$  and an adequate system of local coordinates  $(x, y, z)$  centered at  $o$ ,  $(N, o) \subset \{xy = 0\}$  and we can consider a parametrization of  $(S, o)$  that is in strong normal form.  $\square$

**Theorem 5.3.** *Let  $M$  be a manifold of dimension 3. Let  $S$  be a quasi-ordinary surface of  $M$  at  $o$ . Let  $N$  be a general configuration adapted to  $S$  at  $o$ . Let  $\Sigma$  be the logarithmic limit of tangents of  $S$  relatively to  $N$ .*

(a) *Assuming that  $p + \zeta$  is in normal form:*

(i) *If  $T_o N = \emptyset$ ,  $\Sigma$  is trivial if and only if*

( $\phi 1$ )  $\mu \geq 1$  or

- ( $\phi 2$ )  $\mu = 0, b \geq 1$ .
- (ii) If  $T_o N = T_o\{x = 0\}$ ,  $\Sigma$  is trivial if and only if
- (x1)  $\mu \geq 1$  or
- (x2)  $\mu = 0$  and  $b \geq 1$ .
- (iii) If  $T_o N = T_o\{y = 0\}$ ,  $\Sigma$  is trivial if and only if
- (y1)  $\lambda \geq 1$ .
- (iv) If  $T_o N = T_o\{z = 0\}$ ,  $\Sigma$  is trivial if and only if
- (z1)  $\mu = 0, b \geq 1$ .
- (v) If  $T_o N = T_o\{xz = 0\}$ ,  $\Sigma$  is trivial if and only if
- (xz1)  $\mu \neq 0$  or
- (xz2)  $\mu = 0, b \geq 1$  or
- (xz3)  $\mu = 0, b < 1, a = \lambda$ .
- (vi)  $\Sigma$  is trivial if  $T_o N = T_o\{xy = 0\}$ ,  $T_o N = T_o\{yz = 0\}$  or  $T_o N = T_o\{xyz = 0\}$ .
- (b) Let  $S$  be the germ of a quasi-ordinary surface at  $o$  with trivial limit of tangents. Set  $\Gamma = \mathbb{P}_S^*\langle M/N \rangle$ . Let  $L$  be one of the admissible centers for  $S$  considered in Table 1. Set  $\Lambda = \mathbb{P}_L^*\langle M/N \rangle$ . Let  $\tilde{\Gamma}$  be the proper inverse image of  $\Gamma$  by the blow up of  $\mathbb{P}^*\langle M/N \rangle$  with center  $\Lambda$ . Then

$$\tilde{\Gamma} = \mathbb{P}_{\tilde{S}}^*\langle M/N \rangle.$$

*Proof.* Let  $S$  be a surface that verifies the conditions of statement (b). Let  $\lambda$  be the only limit of tangents of  $S$  at  $o$ . Let  $\sigma : T_\Lambda \mathbb{P}^*\langle M/N \rangle \rightarrow T_L M$  be the map associated to  $\pi : \mathbb{P}\langle M/N \rangle \rightarrow M$  by Lemma 4.1. By Lemma 4.2 it is enough to verify that

$$(5.7) \quad C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \Lambda$$

holds in order to show that  $\tilde{\Gamma} \subset \Omega$ , where  $\Omega$  is the set referred to in diagram 4.2. We will prove statements (a) and (b) in each of the cases considered in statement (a). The cases where  $\zeta$  does not depend on  $y$  are easy to handle so we omit them. In case (i) the triviality of the limits of tangents was already discussed in [1]. Set  $\theta = \xi dx + \eta dy + \zeta dz = \zeta(dz - p dx - q dy)$ . In case ( $\phi 1$ ),  $L = \{x = y = z = 0\}$ . Moreover, the blow up of  $\mathbb{P}^* M$  along  $\Lambda = \{x = y = z = 0\}$  equals  $\mathbb{P}^*\langle \tilde{M}/E \rangle$ . Hence (5.7) is trivially verified.

In case ( $\phi 2$ )  $L = \{x = z = 0\}$ . Hence  $\Lambda = \{x = z = q = 0\}$  and  $\sigma(\tilde{x}, y, \tilde{z}, p, \tilde{q}) = (\tilde{x}, y, \tilde{z})$ . Since  $\mu = 0$ ,

$$(5.8) \quad a \geq \lambda \geq 1.$$

Since  $z = a_{\lambda 0} x^\lambda + \dots + a_{ab} x^a y^b + \dots$ ,

$$(5.9) \quad q = \frac{\partial z}{\partial y} = x^a y^{b-1} \varepsilon$$

It follows from (5.8) and (5.9) that  $\Gamma$  is contained in a hypersurface  $q^n + \sum_{i=0}^{n-1} a_i q^i = 0$  where  $a_i \in \mathbb{C}\{x, y\}$  and  $a_i \in (x)^{n-i}$ . Hence there are  $\tilde{a}_i \in (\tilde{x})$  such that  $C_\Lambda(\Gamma)$  is contained in an hypersurface  $\tilde{q}^n + \sum_{i=0}^{n-1} \tilde{a}_i \tilde{q}^i = 0$ . Therefore (5.7) holds.

In case (ii)  $\theta = \zeta(dz - p dx/x - q dy) = \xi dx/x + \eta dy + \zeta dz$ .

Assume  $0 < \mu < 1$ . Then  $z = x^\lambda y^\mu \varepsilon_1$  is a parametrization of  $S$  and

$$z = x^\lambda y^\mu \varepsilon_1, \quad p = x^\lambda y^\mu \varepsilon_2, \quad q = x^\lambda y^{\mu-1} \varepsilon_3.$$

is a parametrization of the regular part of  $\Gamma_{reg}$ .

Set  $\beta = \lambda\alpha/(1-\mu)$ , where  $\alpha$  is a positive integer. There are  $A, B \in \mathbb{C}^*$ , and units  $\sigma_i$  of  $\mathbb{C}\{t\}$ ,  $1 \leq i \leq 3$ , such that the map that takes  $t$  into

$$\left( At^\alpha, Bt^\beta, A^\lambda B^\mu t^{\alpha\lambda+\beta\mu} \sigma_1, A^\lambda B^\mu t^{\alpha\lambda+\beta\mu} \sigma_2, A^\lambda B^{\mu-1} t^{\alpha\lambda+\beta(\mu-1)} \sigma_3 \right)$$

is a curve of  $\Gamma$ . Since  $\alpha\lambda + \beta\mu > 0$  and  $\alpha\lambda + \beta(\mu-1) = 0$ , the curve converges to  $(0, 0, 0, 0 : A^\lambda B^{\mu-1} \sigma_3(0) : 1)$  when  $t$  goes to 0. Hence  $\Sigma$  is not trivial.

Assume  $\mu = 0$  and  $b < 1$ . Then  $z = x^\lambda \delta_1 + x^a y^b \varepsilon_1$ ,  $p = x^\lambda \varepsilon_2$ ,  $q = x^a y^{b-1} \varepsilon_3$  define a parametrization of  $\Gamma_{reg}$ . Hence we can repeat the previous argument.

Assume  $\mu \geq 1$ . Hence  $z = x^\lambda y^\mu \varepsilon_1$  defines a parametrization of  $S$  and  $q = x^\lambda y^{\mu-1} \varepsilon_2$  defines a parametrization of a hypersurface that contains  $\Gamma$ . Hence  $\Sigma \subset \{\eta = 0\}$ . By (3.3),  $\Sigma \subset \{\xi = 0\}$ .

If  $\mu = 0$  and  $b \geq 1$  we can obtain a proof of the triviality of  $\Sigma$  combining the arguments of the previous cases.

In case (x1),  $\mu \geq 1$ . If  $L = \{x = y = z = 0\}$ ,  $\Lambda = \{x = y = z = p = 0\}$  and  $\sigma(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, q) = (\tilde{x}, \tilde{y}, \tilde{z})$ . Then  $z = x^\lambda y^\mu \varepsilon_1$ , and  $p = x^\lambda y^\mu \varepsilon_2$ . Since  $\lambda + \mu > 2$ ,  $C_\Lambda(\Gamma) \subset \{\tilde{p} = 0\}$ . Hence, (5.7) holds.

If  $L = \{y = z = 0\}$ ,  $\Lambda = \{y = z = p = 0\}$ . If  $\mu > 1$ , the argument is similar to the previous one.

If  $\mu = 1$ , then  $z = x^\lambda y \varepsilon_1$ ,  $p = x^\lambda y \varepsilon_2$ , and  $C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \{\tilde{p} = 0\}$ .

(x2) Since  $L = \{x = z = 0\}$ ,  $\Lambda = \{x = z = p = q = 0\}$ . Hence

$$z = x^\lambda \varepsilon_1 = x^\lambda \delta + x^a y^b \varepsilon_2, \quad p = x^\lambda \varepsilon_3, \quad q = x^a y^{b-1} \varepsilon_4.$$

Since  $\mu = 0$ ,  $\lambda > 1$ . Therefore  $C_\Lambda(\Gamma) \subset \{\tilde{p} = 0\}$ .

Since  $a \geq \lambda > 1$ ,  $b \geq 1$ . Therefore  $C_\Lambda(\Gamma) \subset \{\tilde{q} = 0\}$ .

(iii) By (3.3),  $\Sigma \subset \{\eta = 0\}$ . If  $\mu = 0$  and  $\lambda > 1$ ,  $\Sigma \subset \{\xi = 0\}$  by the arguments of case (ii). The same arguments hold if  $\lambda \geq 1$  and  $\mu > 0$ . If  $\lambda < 1$ , the argument of the first case considered in (ii) shows that  $\Sigma$  is not trivial. The proof of (b) is similar to the one presented in case (ii). (iv) Set  $\theta = \xi dx + \eta dy + \zeta dz/z = \zeta(dz/z - p dx - q dy) = \xi(dx - r dy - s dz/z)$ .

Assume that  $\mu \neq 0$ . Then

$$z = x^\lambda y^\mu \varepsilon_1, \quad p = x^{\lambda-1} y^\mu \frac{\varepsilon_2}{z}, \quad q = x^\lambda y^{\mu-1} \frac{\varepsilon_3}{z}$$

is a parametrization of  $\Gamma_{reg}$ . Hence there is a curve on  $\Gamma$  of the type

$$t \mapsto \left( At^\alpha, Bt^\alpha, A^\alpha B^\alpha t^{\alpha(\lambda+\mu)} \sigma_1, \frac{\sigma_2}{At^\alpha}, \frac{\sigma_3}{Bt^\alpha} \right),$$

where  $\sigma_i$ ,  $1 \leq i \leq 3$  are units of  $\mathbb{C}\{t\}$ . Since  $(\sigma_2(At^\alpha)^{-1} : \sigma_3(Bt^\alpha)^{-1} : 1) = (B\sigma_1 : A\sigma_2 : ABt^\alpha)$  converges to  $(B\sigma_1(0) : A\sigma_2(0) : 0)$ ,  $\Sigma = \{\zeta = 0\}$ .

Assume that  $\mu = 0$  and  $b < 1$ . Setting  $\beta = \alpha(\alpha+1-\lambda)(1-b)^{-1}$ , we can show that  $\Sigma = \{\zeta = 0\}$ .

Assume that  $\mu = 0$  and  $b \geq 1$ . Then  $z = x^\lambda \delta + x^a y^b \varepsilon_1 = x^\lambda \varepsilon_2$ ,  $p = \varepsilon_3/x$ ,  $q = x^{a-\lambda} y^{b-1} \varepsilon_4$  define a parametrization of  $\Gamma_{reg}$ . Moreover,  $\Gamma$  is contained in the hypersurfaces defined by the equations

$$(5.10) \quad x\xi + \varepsilon_4 \zeta = 0, \quad \eta + x^{a-\lambda} y^{b-1} \varepsilon_5 \zeta = 0.$$

Hence  $\Sigma = \{\eta = \zeta = 0\}$ .

Let us assume that  $\mu = 0$ ,  $b \geq 1$ , and prove (b). Set  $L = \{x = z = 0\}$ . Hence  $\Lambda = \{x = z = r = s = 0\}$ . From (5.10),  $\Gamma$  is contained in the hypersurfaces with

parametrizations given by  $s = \varepsilon_4^{-1}x, r + x^{a-\lambda}y^{b-1}\varepsilon_5s = 0$ . Hence  $C_\Lambda(\Gamma) \cap \{\tilde{x} = \tilde{z} = 0\} \subset \{\tilde{r} = \tilde{s} = 0\}$ .

(v) Set  $\theta = \zeta(dz/z - pdx/x - qdy) = \eta(dy - rdx/x - sdz/z)$ .

The case  $\mu = 0$  and  $b \geq 1$  is quite similar to the one in (iv).

Assume that  $\mu = 0, b < 1, a = \lambda$ . Then  $z = x^\lambda \varepsilon_1, p = \varepsilon_2, q = y^{b-1} \varepsilon_3$  defines a parametrization of  $\Gamma_{reg}$ . Hence  $\Gamma$  is contained in the analytic set  $\xi + \varepsilon_2 \zeta = 0, y^{1-b} \eta + \varepsilon_3 \zeta = 0$ . Therefore  $\Sigma = \{\xi = \zeta = 0\}$ .

Assume that  $\mu = 0, b < 1, a > \lambda$ . Setting  $\beta = \alpha(a - \lambda)/(1 - b)$ , it can be shown by the previous methods that there is a  $u \in \mathbb{C}^*$  such that  $\Sigma \supset \{(u : v : 1) : v \in \mathbb{C}^*\}$ .

We now consider statement (b) for the case  $\mu = 0$ . We have that  $L = \Lambda = \{x = z = 0\}$ . This situation is solved by theorem 4.3.

Assume  $\mu \neq 0$ . Set  $L = \{x = y = z = 0\}$ . Hence  $\Lambda = \{x = y = z = r = s = 0\}$ . Then  $z = x^\lambda y^\mu \varepsilon_1, p = x^\lambda y^\mu z^{-1} \varepsilon_2 = \varepsilon_3, q = x^\lambda y^{\mu-1} z^{-1} \varepsilon_4 = (\varepsilon_5 y)^{-1}$  is a parametrization of  $\Gamma_{reg}$ , and  $\Gamma \subset \{\xi + \varepsilon_3 \zeta = y \varepsilon_5 \eta + \zeta = 0\}$ . Hence  $r = -\varepsilon_3 s, s = y \varepsilon_5$ . Hence  $C_\Lambda(\Gamma) \cap \rho^{-1}(L) \subset \{\tilde{r} = \tilde{s} = 0\}$ .

If  $L = \{y = z = 0\}$ , then  $\Lambda = \{y = z = r = s = 0\}$ , and this case is solved in a similar fashion.

(vi) If  $N = \{xy = 0\}$ , arguments previously used show that  $\Sigma$  is trivial.

Set  $\theta = \xi dx/x + \eta dy/y + \zeta dz = \zeta(dz - pdx/x - qdy/y)$ . Set  $L = \{x = y = z = 0\}$ .

Hence  $\Lambda = \{x = y = z = p = q = 0\}$ .

Then  $z = x^\lambda y^\mu \varepsilon_1, p = z^\lambda y^\mu \varepsilon_2 = z \varepsilon_3, q = x^\lambda y^\mu \varepsilon_4 = z \varepsilon_5$ . Hence  $\Gamma$  is contained in the hypersurfaces  $p^k + \sum_{i=0}^{k-1} a_i p^i = 0, q^l + \sum_{i=0}^{l-1} b_i q^i = 0$ , where  $a_i \in (z^{k-i}), b_i \in (z^{l-i})$ . Hence (5.7) holds.

Assume that  $\mu = 0, L = \{x = z = 0\}$  or  $\mu \geq 1, L = \{y = z = 0\}$ . In both cases  $C_\Lambda(\Gamma) \subset \{\tilde{p} = \tilde{q} = 0\}$  by the standard arguments.

In the case  $N = \{yz = 0\}$ ,  $\Sigma$  is always trivial by the arguments of case  $N = \{xz = 0\}$ .

If  $\mu \neq 0$ , we are in the situation of (xz1).

Assume that  $\mu = 0$ . Set  $\theta = \xi dx + \eta dy/y + \zeta dz/z = \xi(dx - rdy/y - sdz/z)$ .

Then  $z = x^\lambda \varepsilon_1 = x^\lambda \delta + x^a y^b \varepsilon_2, p = x^{\lambda-1} z^{-1} \varepsilon_3 = (x \varepsilon_4)^{-1}, q = x^a y^b z^{-1} \varepsilon_5 = x^{a-\lambda} y^b \varepsilon_6$ , define a parametrization of  $\Gamma_{reg}$ . Therefore  $\Gamma \subset \{x \varepsilon_4 \xi + \zeta = \eta + x^{a-\lambda} y^b \varepsilon_6 \zeta = 0\}$ . Hence  $\Gamma$  is contained in the hypersurfaces

$$(5.11) \quad s = x \varepsilon_4, r + x^{a-\lambda} y^b \varepsilon_6 s = 0.$$

It follows from (5.11) that  $C_\Lambda(\Gamma) \cap \sigma^{-1}(L) \subset \{\tilde{r} = \tilde{s} = 0\}$  if  $L = \{x = y = z = 0\}$  or  $L = \{x = z = 0\}$ .

If  $N = \{xyz = 0\}$ ,  $\theta = \xi dx/x + \eta dy/y + \zeta dz/z = \zeta(dz/z - pdx/x - qdy/y)$ .

Assume that  $\mu \neq 0$ .  $\Gamma$  is contained in the hypersurfaces determined by  $z = x^\lambda y^\mu \varepsilon_1, p = \lambda x^\lambda y^\mu \varepsilon_2 z^{-1}$  and  $q = \mu x^\lambda y^\mu \varepsilon_3 z^{-1}$ , where  $\varepsilon_1(0) = \varepsilon_2(0) = \varepsilon_3(0)$ . Hence  $\Sigma = \{(\lambda : \mu : 1)\}$ . A similar argument shows that we arrive to the same conclusion when  $\mu = 0$ .

In the case  $N = \{xyz = 0\}$ , (b) is trivially verified. □

**Example 5.4.** Given  $\lambda > 2$  and  $0 < b < 1$ , the surface  $S$  with parametrization  $z = x^\lambda + x^\lambda y^b$  verifies the condition (xz3) of Theorem 5.3. Hence its logarithmic limits of tangents relatively to the divisor  $\{xz = 0\}$  is trivial. The proper inverse

Divisor	Conditions	Label	Center
$\emptyset$	$\mu \geq 1$	$\phi 1$	$\sigma_0$
	$\mu = 0$ and $b \geq 1$	$\phi 2$	$\sigma_x$
$\{x = 0\}$	$\mu \geq 1$	x1	$\sigma_0, \sigma_y$
	$\mu = 0$ and $b \geq 1$	x2	$\sigma_x$
$\{y = 0\}$	$\lambda \geq 1$ and $\mu \neq 0$	y1	$\sigma_x, \sigma_0$
$\{z = 0\}$	$\mu = 0$ and $b \geq 1$	z1	$\sigma_x$
$\{xz = 0\}$	$\mu \neq 0$	xz1	$\sigma_0$
	$\mu = 0$ and $b \geq 1$	xz2	$\sigma_x$
	$\mu = 0, b < 1$ , and $a = \lambda$	xz3	$\sigma_x$
	$\mu \geq 1$	xz4	$\sigma_y$
$\{xy = 0\}$		xy1	$\sigma_0$ if $\lambda < 1$ or $\mu = 0$ .
		xy2	$\sigma_x$ if $\lambda \geq 1$ .
		xy3	$\sigma_y$ if $\mu \geq 1$ .
$\{yz = 0\}$		yz1	$\sigma_0$ if $\lambda < 1$ .
		yz2	$\sigma_0, \sigma_x$ if $\lambda \geq 1$ .
$\{xyz = 0\}$		xyz1	$\sigma_0$
		xyz2	$\sigma_x$ if $\lambda \geq 1$ .
		xyz3	$\sigma_y$ if $\mu \geq 1$ .

TABLE 1.  $\sigma_0 = \{x = y = z = 0\}$ ,  $\sigma_x = \{x = z = 0\}$  and  $\sigma_y = \{y = z = 0\}$ .

image of  $S$  by the blow up with center  $\{x = y = z = 0\}$  admits the parametrization

$$\frac{z}{x} = x^{\lambda-1} + x^{\lambda+b-1} \frac{y}{x}^b.$$

By theorem 5.3, the logarithmic limit of tangents of  $\tilde{S}$  relatively to the divisor  $\{x \frac{z}{x} = 0\}$  is not trivial.

Example 5.4 shows that the triviality of limits of tangents is not hereditary by blowing up. Lemma 5.5 solves this problem.

**Lemma 5.5.** *Let  $N$  be the normal crossings divisor of a germ of manifold  $(M, o)$  of dimension three. Let  $S$  be a quasi-ordinary surface of  $M$  such that the logarithmic limit of tangents of  $S$  along  $N$  is trivial. Let  $\pi : \tilde{M} \rightarrow M$  be the blow up of  $M$  along an admissible center for  $S$  and  $N$ . Let  $E$  be the exceptional divisor of  $\pi$ . Let  $p \in \tilde{S} \cap E$ . If  $S, N$  do not verify condition (xz3) of table (1) at  $o$ ,  $\tilde{S}$  has trivial logarithmic limit of tangents along  $\tilde{N}$  at  $p$  and  $\tilde{S}, \tilde{N}$  do not verify condition (xz3) at  $p$ .*

*Proof.* We will denote by (xy), (yz), (xyz) the situations (xyi), (yzi), (xyzi) for each i. We will assume that  $b \neq +\infty$ . The cases where  $b = +\infty$  are much simpler. We also assume that after a blow up, the surface is not yet smooth.

( $\phi 1$ ) We can assume that  $z = x^\lambda y^\mu \varepsilon_1$ .

On the chart  $(x, \frac{y}{x}, \frac{z}{x})$ ,  $\tilde{S} \cap \{x = 0\} = \{\frac{z}{x} = x = 0\}$  and

$$\frac{z}{x} = x^{\lambda+\mu-1} \frac{y}{x}^\mu \varepsilon_2.$$

Since  $\lambda \geq \mu \geq 1$ , we are in situation (x2) at each point of  $\tilde{S} \cap \{x = 0\}$ . The same happens in the chart  $(\frac{x}{y}, y, \frac{z}{y})$ .

( $\phi 2$ ) We can assume that  $z = x^\lambda \delta_1 + x^a y^b \varepsilon_1$ . On the chart  $(x, y, \frac{z}{x})$ ,  $\tilde{S} \cap \{x = 0\} = \{\frac{z}{x} - a = x = 0\}$ ,  $a = 0$  if  $\lambda > 1$ ,  $a \in \mathbb{C}^*$  otherwise, and

$$\frac{z}{x} = x^{\lambda-1} \delta_2 + x^{a-1} y^b \varepsilon_2.$$

If  $\lambda > 2$ , we are in situation (x2) at each point of  $\tilde{S} \cap \{x = 0\}$ . Assume  $1 < \lambda < 2$ . Following a generalization of the proof of Theorem 3.5.5 of [12],  $\tilde{S}$  admits the parametrization

$$x = (\frac{z}{x})^{\frac{1}{\lambda-1}} \delta_3 + (\frac{z}{x})^{\frac{a}{\lambda-1}-1} y^b \varepsilon_3.$$

Hence we are in situation (z1) at each point of  $\tilde{S} \cap \{x = 0\}$ . If  $\lambda = 1$ , the situation is analogous to  $\lambda > 2$  or  $1 < \lambda < 2$ .

(z1) The reasoning is similar to ( $\phi 2$ ) and we are in situation (xz2) or (x2) or (z1) at all points of  $\tilde{S} \cap E$ .

(xz1) Assume that  $\lambda + \mu > 1$ . On the chart  $(x, \frac{y}{x}, \frac{z}{x})$ ,  $\tilde{N} = \{x \frac{z}{x} = 0\}$ ,  $\tilde{S} \cap \{x = 0\} = \{x = \frac{z}{x} = 0\}$  and

$$\frac{z}{x} = x^{\lambda+\mu-1} \frac{y}{x} \varepsilon_2.$$

At the origin of the chart, if  $\lambda > 1$  we are in situation (xz1) or (xz4), otherwise we are in situation (yz) or (xz1) or (xz4). At a point of  $\tilde{S} \cap \{x = 0\}$  where  $\frac{y}{x} \neq 0$ , the reasoning is similar to ( $\phi 2$ ) and we are in situation (xz2). On the chart  $(\frac{x}{y}, y, \frac{z}{y})$ ,  $\tilde{N} = \{\frac{x}{y} y \frac{z}{y} = 0\}$ ,  $\tilde{S} \cap \{y = 0\} = \{y = \frac{z}{y} = 0\}$  and

$$\frac{z}{y} = \frac{x}{y}^\lambda y^{\lambda+\mu-1} \varepsilon_2.$$

Hence, we are in situation (xyz) at the origin of the chart. At a point of  $\tilde{S} \cap \{y = 0\}$  where  $\frac{x}{y} \neq 0$ , we are in situation (xz2)

Assume that  $\lambda + \mu = 1$ . On the chart  $(x, \frac{y}{x}, \frac{z}{x})$ , after the change of coordinates  $\tilde{x} = \frac{z}{x}, \tilde{y} = x, \tilde{z} = \frac{y}{x}$ ,  $\tilde{N} = \{\tilde{x} \tilde{y} = 0\}$  and

$$\tilde{z} = \tilde{x}^{\frac{1}{\mu}} \varepsilon_3.$$

We are in situation (xy2) at  $(0, 0, 0)$ . At a point of  $\tilde{S} \cap \{\tilde{y} = 0\}$  where  $\tilde{x} \neq 0$ , we are in situation (x2) or (z1). The reasoning for the chart  $(\frac{x}{y}, y, \frac{z}{y})$  is analogous.

Assume that  $\lambda + \mu < 1$ . On the chart  $(x, \frac{y}{x}, \frac{z}{x})$ ,  $\tilde{N} = \{x \frac{z}{x} = 0\}$ ,

$$\frac{y}{x} = (\frac{z}{x})^{\frac{1}{\mu}} x^{\frac{1-\lambda-\mu}{\mu}} \varepsilon_2$$

and  $\tilde{S} \cap \{x = 0\} = \{\frac{y}{x} = x = 0\}$ . At the origin of the chart we are in situation (xz1) or (xz4). At the remaining points of  $\tilde{S} \cap \{x = 0\}$ , the reasoning is similar to the case ( $\phi 2$ ) and we are in situation (x2) or (z1). The reasoning on the chart  $(\frac{x}{y}, y, \frac{z}{y})$  is similar and the possible situations are (xyz) and (xz2).

In the chart  $(\frac{x}{z}, \frac{y}{z}, z)$ ,  $\tilde{N} = \{\frac{x}{z} z = 0\}$  and

$$z = (\frac{x}{z})^{\frac{\lambda}{1-\lambda-\mu}} (\frac{y}{z})^{\frac{\mu}{1-\lambda-\mu}} \varepsilon_2.$$

We are in the situation (xz1) or (xz4) or (yz) at  $(0, 0, 0)$ . Let  $p$  be a point of  $\tilde{S} \cap \{z = 0\}$ . If  $\frac{x}{z} \neq 0, \frac{y}{z} = 0$  at  $p$ , the reasoning is similar to ( $\phi 2$ ) and we are in situation (x2) or (z1). The reasoning is similar if  $\frac{x}{z} = 0, \frac{y}{z} \neq 0$  at  $p$  and we are in situation (xz2). If  $\frac{x}{z}, \frac{y}{z} \neq 0$  at  $p$ ,  $\tilde{S}$  is smooth at  $p$ .

(xz2) The reasoning is similar to ( $\phi 2$ ) and we are in situation (xz2) or (x2) or (z1) at all points of  $\tilde{S} \cap E$ .



(xz4) We can assume that  $z = x^\lambda y^\mu \delta_1 + x^a y^b \varepsilon_1$ . On the chart  $(x, y, \frac{z}{y})$ ,  $\tilde{N} = \{xy\frac{z}{y} = 0\}$  and

$$\frac{z}{y} = x^\lambda y^{\mu-1} \delta_2 + x^a y^{b-1} \varepsilon_2.$$

Hence we are in situation (xyz) at the origin. At the remaining points of  $\tilde{S} \cap \{y = 0\}$  we are in situation (xz2), if  $\mu > 1$ , and in situation (x2), if  $\mu = 1$ .

The remaining cases are similar to those studied in this proof.  $\square$

**Theorem 5.6.** *Let  $S$  be a quasi-ordinary surface of a germ of manifold of dimension 3,  $(M, o)$ . Assume that the limit of tangents of  $S$  at  $o$  is trivial. Let  $M_0 = M, \Gamma = \mathbb{P}_S^* M$ . Let*

$$M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots \leftarrow M_m$$

*be the sequence of blow ups that desingularizes  $S$ . Let  $L_i$  be the center of the blow up  $M_{i+1} \rightarrow M_i$  for  $0 \leq i \leq m-1$ . Let  $S_i$  be the proper inverse image of  $S$  by the map  $M_i \rightarrow M_0$ . Let  $N_i$  be the inverse image of the first center by the map  $M_i \rightarrow M_0$ . Set  $\Gamma_i = \mathbb{P}_{S_i}^* \langle M_i/N_i \rangle$ ,  $\Lambda_i = \mathbb{P}_{L_i}^* \langle M_i/N_i \rangle$ . Let  $X_i$  be the blow up of  $\mathbb{P}^* \langle M_i/N_i \rangle$  along  $\Lambda_i$ . There are inclusion maps  $\mathbb{P}^* \langle M_{i+1}/N_{i+1} \rangle \hookrightarrow X_i$  such that the diagram (5.12) commutes.*

$$(5.12) \quad \begin{array}{ccccccc} \mathbb{P}^* M_0 & \leftarrow & \mathbb{P}^* \langle M_1/N_1 \rangle & \leftarrow & \cdots & \leftarrow & \mathbb{P}^* \langle M_m/N_m \rangle \\ \downarrow & & \downarrow & & & & \downarrow \\ M_0 & \leftarrow & M_1 & \leftarrow & \cdots & \leftarrow & M_m \end{array}$$

*Moreover  $\Gamma_m$  is a regular Lagrangean variety transversal to the set of poles of  $\mathbb{P}^* \langle M_m/N_m \rangle$  and  $\Gamma_m$  is the proper inverse image of  $\Gamma_0$  by the map  $\mathbb{P}^* \langle M_m/N_m \rangle \rightarrow \mathbb{P}^* M$ .*

*Proof.* It follows from [2], Theorems 4.3, 4.4 and 5.3 and Lemma 5.5.  $\square$

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